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Calculation of the radiation force on a cylinder in a standing wave acoustic field

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Abstract

We present a new calculation of the radiation force on a cylinder in a standing wave acoustic field. We use the formula to calculate the force on a cylinder which is free to move in the field and one which is fixed in space.

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1. Introduction

The radiation force is the time-averaged force exerted on an object in a sound field. It is a nonlinear effect due to momentum/energy transfer from the harmonic wave to the object [1–6]. Some publications refer to the force on an object as radiation pressure; however, this ‘pressure’ is a vector quantity and usually integrated over the surface area of the object, so we will use radiation force.

When an object is placed in a standing wave acoustic field a series of four maxima and minima in the force are created, which cause free objects to be driven to one of the minima. The specific minima (pressure anti-node or velocity anti-node) are determined by the density and compressibility of the object relative to the fluid [4, 7]. The radiation force can be generated by a non-zero time-averaged pressure [8], drag due to acoustic streaming [8], the Reynolds stresses due to the deformation of the surface of the object [8], and a contribution due to the dynamics of an object in the acoustic field [9].

In this paper, as is the common practice, we consider the case where the dimensions of the object are much greater than the acoustic boundary layer thickness. Here the fluid can be assumed to be inviscid and the effects of streaming are neglected [8, 10]. King was one of the first to analyse the radiation force [4]. He published a landmark paper describing the radiation force on a sphere due to wave propagation in an inviscid fluid. He derived a formula for the second-order pressure and calculated the radiation force due to a standing wave and a travelling wave. Most studies presented have only considered spherical objects, but there have been a few investigations into the radiation force on a cylinder. These considered an inviscid

fluid where the cylinder is free to move in the acoustic field. Awatani was probably the first to calculate the radiation force on a cylinder in 1954 [11]. He presented calculations for the variation in the force on a rigid cylinder due to a travelling wave field. He also claimed to have calculated the radiation force for a standing wave but on inspection the wave propagates in the x -direction and therefore is in fact not a standing wave. In 1988 and 1993 Hasegawa *et al* [12, 13] published calculations for elastic cylinder spherical shells and cylindrical shells in a travelling wave field. They did not consider a standing wave acoustic field. In 1990 Wu *et al* also produced an analytical study which was compared to experimental results [14]. They found an agreement to within 20%. However, their calculation was for two incident waves rather than an incident and scattered wave. They do not consider the variation of the wave amplitude over the surface of the cylinder so only solve the boundary conditions on the x -axis for a cylinder of radius $r = 0$, and the wave is assumed to be symmetric about the centre of the object. When this method is compared with more complete solutions the errors can be significant. Edenezer and Stepanishen presented two papers using numerical solutions to the radiation force for a cylinder that is vibrating at an arbitrary number of natural frequencies [15, 16]. They considered the flutter produced by a steady flow field but did not consider an acoustic field incident on the cylinder. None of these studies satisfactorily analyses the case of a cylinder in a standing wave field.

Here we present a new calculation of the radiation force for a rigid cylinder that is free to move in the acoustic field in an inviscid fluid. The radiation force for a cylinder fixed in space can also be calculated by setting the particle velocity $v_p = 0$. In a further paper and a D Phil thesis [17, 18] we will use these calculations to show that lattice Boltzmann simulations can model the radiation force. We now describe the calculation of the radiation force on a cylinder. In section 3 we present an example calculation, then in section 4 we present our conclusions.

2. Calculation of the radiation force on a rigid cylinder in an inviscid fluid

We assume that the acoustic field does not deform the object and the acoustic boundary layer thickness $\beta^{-1} = \sqrt{2\nu/\omega} \ll a$ where ν is the kinematic viscosity, ω is the angular frequency, and a is the radius of the cylinder. Under these conditions the fluid can be treated as inviscid [10] and the radiation force is given by the time average of the pressure on the surface of the object [4]

$$F_\alpha = - \oint \langle P \rangle \hat{n}_\alpha \, dA \quad (1)$$

where the pressure P is calculated on the surface of the particle A . For a fixed particle [4, 19]

$$P - P_0 = P' = -\rho_0 \dot{\phi} + \frac{1}{2} \frac{\rho_0}{c^2} \dot{\phi}^2 - \frac{1}{2} \rho_0 u^2 \quad (2)$$

where $\mathbf{u} = \nabla\varphi$ is the velocity of the fluid, φ is the velocity potential for compressible irrotational flow, ρ_0 is the undisturbed density of the fluid, P_0 is the undisturbed pressure and c is the speed of sound in the fluid. If the particle is free to move due to the action of the wave we need to transform (2) into the Lagrangian reference frame. To do this the first term of (2) is transformed as follows [4]

$$\dot{\phi} = D_t \varphi - \dot{\xi} \partial_x \varphi - \dot{\eta} \partial_y \varphi - \dot{\zeta} \partial_z \varphi = D_t \varphi - \dot{\xi} u_x - \dot{\eta} u_y - \dot{\zeta} u_z \quad (3)$$

where $\dot{\xi}$, $\dot{\eta}$, $\dot{\zeta}$ are the transformation velocities of the particle and co-ordinate system, and

$$\langle D_t \varphi \rangle = 0. \quad (4)$$

We can now consider the radiation force as being the sum of the contributions from the time-average potential energy $\langle P_\phi \rangle$, kinetic energy $\langle P_q \rangle$ and a contribution due to the motion of the particle $\langle P_\zeta \rangle$ [9]

$$F_x = \langle P_\phi \rangle + \langle P_q \rangle + \langle P_\zeta \rangle. \tag{5}$$

The motion of the particle $\mathbf{v}_p = (\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is found from the acceleration due to first-order contribution of the pressure field on its surface [4, 11] which after integrating with respect to time gives

$$m\mathbf{v}_p = \oint \rho_0 \varphi \hat{\mathbf{n}} dA \tag{6}$$

where m is the mass of the particle. The motion of the particle will influence the acoustic field so it must be incorporated as a boundary condition when determining the scattered velocity potential.

For a standing wave propagating in the x -direction the motion of an infinitely long cylinder is given by

$$v_{px} = \frac{\rho_0}{\rho_1} \frac{1}{\pi a} \int_0^{2\pi} \varphi \cos \theta d\theta = \dot{\xi} = \text{Re}[v_{pfx} e^{i\omega t}]. \tag{7}$$

where v_{pfx} is the time-independent velocity of the cylinder and ρ_1 is its density.

For a standing wave in the cylindrical co-ordinates with the origin at the centre of the cylinder the time-independent incident and scattered velocity potentials are given by

$$\varphi_{fi} = V'_0 \cos \psi \tag{8}$$

$$\varphi_{fs} = \sum_0^\infty H_n^{(2)}(kr) [A'_n \cos(n\theta) + B'_n \sin(n\theta)] \tag{9}$$

where $\psi = k(r \cos \theta + h)$, h is the distance in the x -direction from the source of the acoustic wave to the centre of the cylinder, $V'_0 = V_0/k$, V_0 is the velocity amplitude at the source, k is the wave vector ($k = \omega/c$), φ_{fi} and φ_{fs} are the time-independent incident and scattered velocity potentials respectively, $H_n^{(1)}(kr)$ and $H_n^{(2)}(kr)$ are n th-order Hankel functions of the first and second kind respectively [20], and A'_n and B'_n are constants which are defined from the boundary conditions on the surface of the cylinder. The total velocity potential is $\varphi = \text{Re}[\varphi_f e^{i\omega t}] = \text{Re}[(\varphi_{fi} + \varphi_{fs}) e^{i\omega t}]$.

To perform the required integrals to solve for the radiation force we use power series expansions of $\cos \psi$ and $\sin \psi$

$$\cos \psi \approx 1 - \frac{\psi^2}{2!} + \frac{\psi^4}{4!} - \frac{\psi^6}{6!} + \frac{\psi^8}{8!} - \dots \tag{10}$$

$$\sin \psi \approx \psi - \frac{\psi^3}{3!} + \frac{\psi^5}{5!} - \frac{\psi^7}{7!} + \dots \tag{11}$$

For our studies we consider maximum $\psi \sim 0.75\pi$ which allows us to discard higher order terms than $\psi^8/8!$ as $(0.75\pi)^8 \ll 8!$. We also work in the limit $ka \ll kh$ and are only interested in solutions to φ for $r \approx a$ which allows us to only consider terms up to $O((kr)^2)$.

Using the boundary condition $u_{fr} = v_{pfx} \cos \theta$ at $r = a$, and taking a Fourier transform of (8) after substituting (10) we obtain the time-independent velocity potential

$$\varphi_f = V'_0 \left[(A_r + B_r \cos \theta + C_r \cos^2 \theta) + \frac{E_a}{2} \frac{H_0^{(2)}(kr)}{H_0^{(2)'}(ka)} + \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V'_0 k} \right) \frac{H_1^{(2)}(kr)}{H_1^{(2)'}(ka)} \cos \theta + \frac{E_a}{2} \frac{H_2^{(2)}(kr)}{H_2^{(2)'}(ka)} \cos(2\theta) \right] \tag{12}$$

$$\varphi = \text{Re}[\varphi_f e^{i\omega t}] \quad (13)$$

where

$$\begin{aligned} A_r &= 1 - \frac{(kh)^2}{2!} + \frac{(kh)^4}{4!} - \frac{(kh)^6}{6!} + \frac{(kh)^8}{8!}, \\ B_r &= -(kh)(kr) + \frac{(kh)^3(kr)}{3!} - \frac{(kh)^5(kr)}{5!} - \frac{(kh)^7(kr)}{7!}, \\ C_r &= -\frac{(kr)^2}{2!} + \frac{6(kh)^2(kr)^2}{4!} - \frac{15(kh)^4(kr)^2}{6!} + \frac{28(kh)^6(kr)^2}{8!}, \\ D_a &= kh - \frac{(kh)^3}{3!} + \frac{(kh)^5}{5!} - \frac{(kh)^7}{7!}, \\ E_a &= ka - \frac{(kh)^2(ka)}{2!} + \frac{(kh)^4(ka)}{4!} - \frac{(kh)^6(ka)}{6!}, \\ F_a &= -\frac{(kh)(ka)^2}{2!} + \frac{2(kh)^3(ka)^2}{4!} - \frac{23(kh)^5(ka)^2}{6!}, \end{aligned}$$

and we define $A_a, B_a,$ and C_a as the values of the constants at $r = a$ and

$$H_n^{(2)'}(kr) = \partial_{kr} H_n^{(2)}(kr).$$

We now substitute (12) into (7) to find the particle velocity. After substitution and integration we find that

$$v_{pfx} = V_0' \frac{B_a + \left(D_a + \frac{3F_a}{4}\right) \frac{H_1^{(2)}(ka)}{H_1^{(2)'}(ka)}}{\frac{\rho_1 a}{\rho_0} - \frac{1}{k} \frac{H_1^{(2)}(ka)}{H_1^{(2)'}(ka)}}. \quad (14)$$

We now find the radiation force from a surface integral of the time-averaged surface contributions of the potential energy, kinetic energy and the contribution due to the motion of the particle.

The contribution due to the time-averaged potential energy ($\langle V \rangle = \frac{1}{2} \rho_0 \dot{\varphi}^2 / c^2$) is given by [4] as

$$\langle P_\phi \rangle = -\frac{a\rho_0}{2c^2} \int_0^{2\pi} \langle \dot{\varphi}^2 \rangle \cos \theta \, d\theta. \quad (15)$$

Inserting (13) into (15) performing the required differentiation and using the relationship

$$\langle \text{Re}[f(\mathbf{x}) e^{i\omega t}] \text{Re}[g(\mathbf{x}) e^{i\omega t}] \rangle = \frac{1}{2} \text{Re}[f(\mathbf{x})g(\mathbf{x})^*], \quad (16)$$

where $g(\mathbf{x})^*$ is the complex conjugate of $g(\mathbf{x})$, and noting that $H_n^{(2)}(kr)^* = H_n^{(1)}(kr)$ and $H_n^{(2)'}(kr)^* = H_n^{(1)'}(kr)$ we find that

$$\langle P_\phi \rangle = -\frac{\pi a \rho_0 (V_0' \omega)^2}{4c_0^2} \text{Re}[f_1(kr)] \quad (17)$$

$$\begin{aligned} f_1(ka) &= \left(2A_a B_a + \frac{3}{2} B_a C_a\right) + \frac{B_a E_a}{2} \frac{H_0^{(1)}(ka)}{H_0^{(1)'}(ka)} \\ &\quad + \left(A_a + \frac{3}{4} C_a\right) \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0' k}\right) \frac{H_1^{(1)}(ka)}{H_1^{(1)'}(ka)} + \frac{B_a E_a}{4} \frac{H_2^{(1)}(ka)}{H_2^{(1)'}(ka)} \end{aligned}$$

$$\begin{aligned}
 & + \frac{B_a E_a}{2} \frac{H_0^{(2)}(ka)}{H_0^{(2)'}(ka)} + \frac{E_a}{2} \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_0^{(2)}(ka) H_1^{(1)}(ka)}{H_0^{(2)'}(ka) H_1^{(1)'}(ka)} \\
 & + \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \left(A_a + \frac{3}{4} C_a \right) \frac{H_1^{(2)}(ka)}{H_1^{(2)'}(ka)} \\
 & + \frac{E_a}{2} \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_1^{(2)}(ka) H_0^{(1)}(ka)}{H_1^{(2)'}(ka) H_0^{(1)'}(ka)} \\
 & + \frac{E_a}{4} \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_1^{(2)}(ka) H_2^{(1)}(ka)}{H_1^{(2)'}(ka) H_2^{(1)'}(ka)} + \frac{B_a E_a}{4} \frac{H_2^{(2)}(ka)}{H_2^{(2)'}(ka)} \\
 & + \frac{E_a}{4} \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_2^{(2)}(ka) H_1^{(1)}(ka)}{H_2^{(2)'}(ka) H_1^{(1)'}(ka)}. \tag{18}
 \end{aligned}$$

The contribution due to the time-averaged kinetic energy $\langle T \rangle = \langle \frac{1}{2} \rho_0 u^2 \rangle$ is given by

$$\langle P_q \rangle = \frac{a \rho_0}{2} \int_0^{2\pi} \langle u_1^2 \rangle \cos \theta \, d\theta \tag{19}$$

where $\mathbf{u}_1 = u_{1\theta} + u_{1r}$, $u_1^2 = u_{1\theta}^2 + u_{1r}^2$, substituting into (19) and using the relationship (16) gives

$$\langle P_q \rangle = \frac{\rho_0}{4a} \operatorname{Re} \int_0^{2\pi} [(\partial_\theta \varphi_f)(\partial_\theta \varphi_f)^* + (v_{pfx})(v_{pfx})^*] \cos \theta \, d\theta. \tag{20}$$

Noting that $(v_{pfx})(v_{pfx})^* \int_0^{2\pi} \cos \theta \, d\theta = 0$ we find

$$\langle P_q \rangle = \frac{\rho_0 \pi V_0'}{8a} \operatorname{Re} f_3(kr) \tag{21}$$

$$\begin{aligned}
 f_3(kr) = & C_a \left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \left[\frac{H_1^{(2)}(ka)}{H_1^{(2)'}(ka)} + \frac{H_1^{(1)}(ka)}{H_1^{(1)'}(ka)} \right] + 2B_a C_a - C_a G_a \\
 & + \left[\left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_1^{(2)}(ka)}{H_1^{(2)'}(ka)} + B_a - \frac{G_a}{2} \right] \frac{E_a H_2^{(1)}(ka)}{H_2^{(1)'}(ka)} \\
 & + \left[\left(D_a + \frac{3}{4} F_a + \frac{v_{pfx}}{V_0'k} \right) \frac{H_1^{(1)}(ka)}{H_1^{(1)'}(ka)} + B_a - \frac{G_a}{2} \right] \frac{E_a H_2^{(2)}(ka)}{H_2^{(2)'}(ka)} \tag{22}
 \end{aligned}$$

where $G_a = ka F_a$.

We now must include the contribution from converting from the Eulerian to the Lagrangian reference frame due to the motion of the particle in the sound field [4]

$$\langle P_\zeta \rangle = -\rho_0 a \int_0^{2\pi} \langle \mathbf{u} \cdot \mathbf{v}_p \rangle \cos \theta \, d\theta. \tag{23}$$

Using the relationship (16) and the boundary conditions $u_{fr} = v_{pfx} \cos \theta$ at the surface of the cylinder (23) can be rewritten as

$$\langle P_\zeta \rangle = \rho_0 a \int_0^{2\pi} \frac{1}{2} \operatorname{Re} \left[-(v_{pfx})(v_{pfx})^* \cos^2 \theta + \frac{v_{pfx}}{a} \sin \theta (\partial_\theta \varphi_f)^* \right] \cos \theta \, d\theta \tag{24}$$

$$= -\frac{\rho_0 \pi}{2} \operatorname{Re} \left[v_{pfx} \left(C_a + E_a \frac{H_2^{(1)}(ka)}{H_2^{(1)'}(ka)} \right) \right] \tag{25}$$

and from (5) the total radiation force is given by $F_x = \langle P_\phi \rangle + \langle P_q \rangle + \langle P_\zeta \rangle$.

3. Example

We take the fluid properties to be those of air ($c = 340 \text{ m s}^{-1}$, $\rho_0 = 1.2 \text{ kg m}^{-3}$, $v = 1.4 \times 10^{-5}$) and consider a $1.4 \text{ }\mu\text{m}$ radius cylinder in a 2.4 MHz standing wave field of intensity $I = 2.4 \text{ kW m}^{-2}$ ($I = 0.24 \text{ W cm}^{-2}$, $\lambda = 0.143 \text{ mm}$, $V_0 = 3.4 \text{ m s}^{-1}$). We take $h = 3\lambda/8$, the second of the maxima in the radiation force.

First we consider a cylinder fixed in space. Here the velocity of the cylinder $\mathbf{v}_p = v_{px} = v_{pfx} = 0$. Substituting this into equations (17) and (21) gives $\langle P_\phi \rangle = -1.932 \times 10^{-6} \text{ N}$, $\langle P_q \rangle = -1.014 \times 10^{-6} \text{ N}$ and therefore the radiation force $F_x = -2.946 \times 10^{-6} \text{ N}$.

We now consider a cylinder that is free to move in the fluid. We consider a cylinder with density $\rho_l = 120 \text{ kg m}^{-3}$. At $h = 3\lambda/8$ equations (17), (21) and (24) now give $\langle P_\phi \rangle = -1.912 \times 10^{-6} \text{ N}$, $\langle P_q \rangle = -1.004 \times 10^{-6} \text{ N}$, $\langle P_\zeta \rangle = 1.93 \times 10^{-8} \text{ N}$ and therefore the radiation force $F_x = -2.896 \times 10^{-6} \text{ N}$. As the cylinder is free to move it will have a time-averaged motion towards $h = \lambda/4$.

4. Conclusions

In this paper we have produced a new calculation on the radiation force on a cylinder in an inviscid fluid where $ka \ll kh$ and $h \leq 0.335\lambda$. We believe that these calculations are more accurate than those previously reported and can easily be evaluated using standard numerical packages such as matlab. We have also presented an example calculation for both a fixed cylinder unable to move in the acoustic field and a cylinder free to move in the field.

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